Q. If we have a "nice" function $f(x)$, how do we construct a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ which converges to $f(x)$ near $x = a$?

Q. First, why do we care?

Q. Okay, now answer the question.

Note:

Ex 1 $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ if $x \in (-1, 1)$

Ex 2 $\dfrac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}$ for $x \in (-1, 1)$

Ex 3 $\ln(x) = \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}(x-1)^n$ for $x \in (0, 2)$

Ex 4 $\arctan(x) = \sum_{m=0}^{\infty} \dfrac{(-1)^m}{2m+1} x^{2m+1}$ for $x \in (-1, 1)$

Q. What makes it possible find power series for these functions? For which "nice" functions doesn't this approach work?

Q. What is a better approach for finding the coefficients of a power series representation of $f(x)$? Suppose $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ for all x in $(a-R, a+R)$.

PQ. What is $f(a)$?

(a) 0 (b) c_0 (c) c_1 (d) a

Def: Suppose f is a function that is _____ at $x = a$ (i.e. $f(a), f'(a), f''(a), f^{(3)}(a), \ldots$ all exist). The _____ for f at a is and the _____ for f at a is

If $a = 0$ it is sometimes called a _____.

"Thm:" If $f(x)$ is "nice," then the Taylor series for $f(x)$ converges to $f(x)$ on its radius of convergence.

"Proof:"

Ex 5 Find the Taylor series for $f(x) = e^x$ centered at 0, and find its interval of convergence.

PQ. Fix an integer $n \geqslant 0$. What is $f^{(n)}(0)$?

(a) 1 (b) $\frac{1}{n!}$ (c) 0 (d) $n!$

Ex 6 Find the Taylor series for $f(x) = \cos(x)$ centered at 0 and find its interval of convergence.

Cool Stuff:

Q. So why is it that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$?

Def: Let $f(x)$ be a smooth function at $x = a$. The kth partial sum of the Taylor Series for f centered at a, namely

is called the _____. The _____ is the function

Thm: (Taylor's Theorem with Remainder) Suppose f is $(k+1)$-times differentiable on an interval $(a-r, a+r)$ with $x \in (a-r, a+r)$. Let I denote the interval $[a, x]$ if $x \geqslant a$ or $[x, a]$ if $x \leqslant a$.

(a) There is a number $c \in I$ such that $R_k(x) = \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}$

(b) If $|f^{(k+1)}(z)| \leqslant M$ for _____ $z \in I$, then

Ex 7 Use the Maclaurin series for $f(x) = e^x$ to estimate (i) e^1, (ii) e^5, (iii) e^{15} each with error < 0.0001.

Summary: (a) If f is smooth at a, then we can compute the Taylor series at a:

(b) Using the ratio test, we can find a value r such that the Taylor series converges for all x in $(a-r, a+r)$.

(c) Given x in $(a-r, a+r)$, Taylor's Theorem tells us:

Observe:

Ex 8 Why is the Maclaurin series for $\cos(x)$ equal to $\cos(x)$?

Ex 9 Find a value of k for which the kth Taylor polynomial for e^x is within 0.001 of e^x for every $x \in (-2, 2)$.